The superfluidity of fermions coupled to gravity

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Abstract

We investigate superfluidity of the relativistic fermi-gas with gravitational interaction. The exciteation spectrum is obtained within the linearized theory. While superfluidity may take place at a definite ratio of the Fermi momentum, rest mass and coupling constant, the metric coefficients play predominant role forming the gap of excitation spectrum.

1 Introduction

The superfluidity (superconductivity, in the strict sense) of relativistic fermi-systems is under scientific attention since Bailin and Love [6] initiated investigation in this field. Several significant papers were dedicated to superfluidity in nuclear systems [2] with direct applications to a neutron star matter. Among them the paper by Kucharek and Ring [1] and, especialy, Ref. [3] and [4] should be noted as the basic works where the general approach to relativistic superfluidity is outlined. Besides, Capelle and Gross [5] have shown how to find the excitation spectrum.

In the present paper we shall discuss superfluidity of a fermi-system with gravitational interaction between particles. This kind of fermi gas attracts the interst in recent years [7].

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In general the investigation of superfluidity includes several steps. 1) Specifying the Lagrangian L or Hamiltonian H; 2) specifying the anomalous terms there; 3) the Bogolubov transformation; 4) obtaining the equations of motion; 5) derivation of the excitation spectrum from equations of motion truncated up to the Hartree approximation.

1.1 The interacting terms

For a given Lagrangian L of the many-fermion interacting system the canonical formalism

$$H = L - \dot{\varphi}_i \pi^i \qquad \pi^i = \frac{\partial L}{\partial \dot{\varphi}_i} \tag{1}$$

allows to find the Hamiltonian

$$H = T + U \tag{2}$$

consisting of the kinetic energy of free fermions

$$T = \int \mathrm{d}^3\mathrm{r}\, \left(i\bar{\psi}\gamma^k \partial_k \psi \, + m \right) \tag{3}$$

and the interaction term

$$U = \int \mathrm{d}^3\mathrm{r}\; \bar{\psi}\Sigma\psi \qquad \Sigma = \; f_i \; \Gamma^i A_i \tag{4}$$

which represents the interaction with field A_i corresponding to vertex Γ^i and coupling constant f_i. If Γ includes no *derivative* coupling we can extract the interacting term immediately from the Lagrangian L.

For instance

$$\Gamma^\omega = \gamma^\mu \qquad \Gamma^\sigma = \mathbf{1} \tag{5}$$

are the vertices of $\sigma - \omega$ model, while the vertex of QCD is

$$\Sigma = f\, \gamma^\mu \lambda_a\, A^a_\mu \tag{6}$$

In order to present the Hamiltonian of great canonical ensemble, the chemical potential μ is introduced as an additional term

$$-\,\mu\, \psi^\dagger \psi \tag{7}$$

Indeed,

$$\int \mathrm{d}^3\mathrm{r}\, \mu\, \psi^\dagger \psi \equiv \mu N \tag{8}$$

This term stands in (4) implicitly, or we can extract $-\mu$ directly from the scalar potential:

$$\Sigma \rightarrow \Sigma - \gamma^0 \mu \tag{9}$$

The Hamiltonian of superfluid fermi-system includes besides (4) the anamalous term [6, 5]

$$W = \bar{\psi}_c\, \beta \Delta \psi + \bar{\psi}_c^\dagger\, \Delta^\dagger \beta \psi^\dagger \qquad \Delta = \ f_i\ \Gamma^i a_i \tag{10}$$

constructed from (4), where a_i is the anamalous field (do not mix it with A_i), while $\bar{\psi}_c = -\psi^T C$ and $\psi_c = C\bar{\psi}^T = C\beta\psi^*$ are the charge-conjugated spinors and

$$C = i \begin{pmatrix} \tau_2 & 0 \\ 0 & \tau_2 \end{pmatrix} \tag{11}$$

is the matrix of charge conjugation. It is clear that

$$-\,C^T = C = \beta C \beta \qquad C^2 = -1 \tag{12}$$

$$C\beta = \beta C = i \begin{pmatrix} \tau_2 & 0 \\ 0 & -\tau_2 \end{pmatrix} \tag{13}$$

and $\bar{\psi}_c \psi$ is replaced by $\varphi_\uparrow \varphi_\downarrow$ in the non-relativistic limit.

1.2 The Bogolubov transformation

Substituting

$$\hat{\psi} = \sum_p \phi_p\,(r)\ \hat{b}_p^\dagger \tag{14}$$

we find a common field equation:

$$\{-i\vec{\gamma} \cdot \nabla + m + \Sigma\}\, \varphi_p = \beta \varepsilon_p \varphi_p \tag{15}$$

(no summation over p in the right side) that is, briefly,

$$\hat{h}\varphi = \varepsilon\varphi \qquad \hat{h} = -i\vec{\alpha} \cdot \nabla + \beta m + \beta\Sigma \tag{16}$$

As soon as the anamalous term (10) appears we must use the Bogolubov transformation

$$\psi = \sum_p u_p\,(r)\;\hat{b}_p^\dagger + v_p^*\,(r)\;\hat{b}_p \qquad |u_p|^2 + |v_p|^2 = 1 \tag{17}$$

instead of (14). For short we shall omit index p; for instance

$$\psi = u\,\hat{b}^\dagger + v^*\,\hat{b} \qquad \psi^\dagger = u^*\hat{b} + v\hat{b}^\dagger \tag{18}$$

Thereby, we get two equations

$$\begin{array}{c} \{-i\vec{\gamma}\cdot\nabla + m + \Sigma\}\,u = \beta\varepsilon u - \Delta^\dagger\beta C v \\ \{-i\vec{\gamma}\cdot\nabla + m + \Sigma\}^\dagger\,v = -\beta\varepsilon v + C\,\beta\Delta u \end{array} \tag{19}$$

for functions u and v instead of one for ϕ. Therefore,

$$\left(\begin{array}{cc} \hat{h}+\Sigma & \beta\Delta^\dagger\beta C \\ -C\,\Delta & \hat{h}+\Sigma \end{array}\right)\left(\begin{array}{c} u \\ v \end{array}\right) = E\left(\begin{array}{c} u \\ v \end{array}\right) \tag{20}$$

Capelle and Gross [5] found two branches of excitations. It is the analogue of the massive and acoustic modes in relativistic bose-condensate [11]. The subject of our interest is the massless mode whose excitation spectrum within the Hartree approximation looks as

$$E^2 = (\varepsilon-\mu)^2 + \|\Delta\|^2 \qquad \varepsilon^2 = \vec{p}^2 + m^2 \tag{21}$$

or

$$E^2\,(\xi) = \xi^2 + \|\Delta\|^2\,(\xi) \tag{22}$$

where the gap

$$\|\Delta\|^2 = -C\,\Delta\beta\Delta^\dagger\beta C \tag{23}$$

is determined immediately from (20). Particularly, for the scalar field we get from (23) the result of Capelle and Gross [5] $\|\Delta\|^2 = |a|^2$.

Discussion in the frames of Bogolubov-Hartree-Fock approximation, for the scalar-vector and pseudoscalar-isovector vertices, is performed in [1, 3]. The basis of exact solution in the frames of density functional theory is not emphasized yet.

2 Coupling with the gravitational field

The Lagrangian of fermions coupled to the gravitational field [8] reads as

$$\Lambda = \sqrt{-g}L \qquad g = \det g_{\mu\nu} \tag{24}$$

where $g_{\mu\nu}$ is the metric tensor and

$$L = i\bar{\psi}\gamma^{\mu}\nabla_{\mu}\psi + m\bar{\psi}\psi \tag{25}$$

has the form of usual free Lagrangian containing, however, gamma-matrices γ^{μ} and covariant derivative

$$\nabla_{\mu} = \partial_{\mu} + \Omega_{\mu} \tag{26}$$

with properties

$$\begin{gathered} \Omega_{\mu} = \tfrac{1}{4}\gamma^{\nu}_{;\mu}\gamma_{\nu} = -\tfrac{1}{4}\Omega_{\mu ab}\Sigma_{ab} \\ \Omega_{\mu ab} = \partial_{\mu}g_{\alpha a}g^{\alpha b} - \Gamma^{\alpha}_{\mu\beta}g_{\alpha a}g^{\beta b} \\ \Sigma_{ab} = \tfrac{1}{2}(\gamma_a\gamma_b - \gamma_b\gamma_a) \qquad \Sigma^2_{ab} = 1 \qquad \gamma^a\Sigma_{ab} = 0 \end{gathered} \tag{27}$$

The Lagrangian includes the nonlinear terms (with respect to field $g_{\mu\nu}$) which differ sufficiently from the usual Lagrangians in flat space whose interacting terms are presented in a simple current-field form (4).

3 The linearized theory

This linearized approximation [8]:

$$\begin{gathered} g_{\alpha a} \approx \eta_{\alpha a} + \tfrac{1}{2}h_{\alpha a} \\ g_{\mu\nu} = g_{\mu a}g_{\nu a} \approx \eta_{\mu\nu} + \tfrac{1}{2}(h_{\mu\nu} + h_{\nu\mu}) \\ \sqrt{-g} \approx 1 + \tfrac{1}{2}h_{aa} = 1 + \tfrac{1}{2}h \\ \Omega_{\mu ab} \approx \tfrac{1}{4}\left\{\partial_{\mu}h_{ba} + \partial_{\mu}h_{ab} - \partial_b h_{\mu a} - \partial_b h_{a\mu} + \partial_a h_{\mu b} + \partial_a h_{b\mu}\right\} \\ \Omega_{\mu} \cong -\tfrac{1}{8}\Sigma_{ab}\partial_{[a}h_{b]\mu} \equiv \tfrac{1}{4}\Sigma_{ab}\partial_b h_{a\mu} \end{gathered} \tag{28}$$

(where $\eta_{\mu\nu}$ is the Minkowsky metric) with respect to weak field $h_{\mu\nu}$ allows to simplify Lagrangian (25) and split

$$L = \left[1 + \tfrac{1}{2}h\right] i\bar{\psi}\gamma^{a}\left[\eta^{\mu}_{a} + h^{\mu}_{a}\right]\left[\partial_{\mu} + \Omega_{\mu}\right]\psi + + \left[1 + \tfrac{1}{2}h\right] m\bar{\psi}\psi \tag{29}$$

into a sum

$$L = L_0 + \tilde{L} \tag{30}$$

of free Lagrangian

$$L_0 = \bar{\psi}\left(i\bar{\psi}\gamma^\mu\partial_\mu + m\right)\psi \tag{31}$$

and the coupling term

$$\tilde{L} = \bar{\psi}\left\{i\left(\frac{1}{2}h\gamma^\mu + \gamma^a h_a^\mu\right)\partial_\mu + i\gamma^\mu\Omega_\mu + \frac{1}{2}hm\right\}\psi \tag{32}$$

4 Hamiltonian of the linearized theory

Having

$$\varphi_i = \{\psi; h_{\mu\nu}\} \tag{33}$$

as dynamical variables, we find Hamiltonian (1) as a sum

$$H = H_0 + U \tag{34}$$

of free Hamiltonian

$$H_0 = \bar{\psi}\left(i\gamma^k\partial_k + m\right)\psi \tag{35}$$

and interacting term

$$U = \bar{\psi}\left\{\tfrac{1}{2}hm + i\left(\tfrac{1}{2}h\,\gamma^k + \gamma^a h_a^k\right)\partial_k + i\gamma^\mu\Omega_\mu\right\}\psi \tag{36}$$

On account of derivative coupling, the term (4) differs from (25).

According to formula (10) the gap matrix is

$$\Delta = i\left(\frac{1}{2}h\gamma^k + \gamma^a h_a^k\right)\partial_k + i\gamma^\mu\Omega_\mu + \frac{1}{2}hm \tag{37}$$

In general the calculation of gap (10) involves tedious arithmetics that can be omitted in several particular cases.

5 Particular Example: a simplified metric of the rotating massive body

For a rotating massive body whose metric is

$$h_a^\mu = \frac{1}{4}h\,\delta_a^\mu \qquad h_0^k = \vec{h} \qquad h, \vec{h} \cong \text{const} \Rightarrow \Omega = 0 \tag{38}$$

the coefficient (28)

$$\Omega_\mu = \frac{1}{4}\Sigma_{0j}\partial_j h_{0\mu} + \frac{1}{4}\Sigma_{ij}\partial_j h_{i\mu} \tag{39}$$

vanishes. Hence, according to (37), the gap matrix is

$$\Delta = \left(\frac{3}{4}h\vec{\gamma} + \gamma^0\vec{h}\right)\cdot\vec{p} + \frac{1}{2}hm \tag{40}$$

and

$$\begin{aligned} \Delta^\dagger &= \left(\tfrac{3}{4}h\vec{\gamma} - \gamma^0\vec{h}\right)\cdot\vec{p} + \tfrac{1}{2}hm \\ \beta\Delta^\dagger\beta &= -\left(\tfrac{3}{4}h\vec{\gamma} + \gamma^0\vec{h}\right)\cdot\vec{p} + \tfrac{1}{2}hm \end{aligned} \tag{41}$$

where

$$\vec{p} \equiv -i\nabla \tag{42}$$

Thereby,

$$\Delta\,\beta\Delta^\dagger\beta = \frac{1}{4}h^2\left[m^2 + \frac{9}{4}\vec{p}^2\right] - \left(\vec{h}\cdot\vec{p}\right)^2 \tag{43}$$

and, finally,

$$\|\Delta\|^2 = -C\,\Delta\beta\Delta^\dagger\beta C = \frac{1}{4}h^2\left[m^2 + \frac{9}{4}\vec{p}^2\right] - \left(\vec{h}\cdot\vec{p}\right)^2 \tag{44}$$

Note that coefficient $\frac{9}{4}$ results from the tensor nature of gravitational coupling: a pure scalar coupling $\bar{\psi}\frac{1}{2}hm\psi$ added to the free Lagrangian (31) leads merely to the gap

$$\|\Delta\|^2 = \frac{1}{4}h^2m^2 \tag{45}$$

which determines the ordinary excitation spectrum of BCS theory [10] with quadratic dependence of excitation energy (22) on ξ.

Substituting the expression

$$(\xi+\mu)^2 - m^2 = \vec{p}^2 \qquad \mu > m \tag{46}$$

in (44), we get

$$\|\Delta\|^2(\xi,\chi) = \frac{1}{4}h^2\left[\frac{9}{4}(\xi+\mu)^2 - \frac{5}{4}m^2\right] - \vec{h}^2\left[(\xi+\mu)^2 - m^2\right]\cos^2\chi \tag{47}$$

Superfluidity takes place if the right side of (47) is positive at any value of ξ.

6 Summary

6.1 Conclusion for a pure spherical metric

Since the Fermi energy $\varepsilon_F \equiv \mu > m$, the gap (47) corresponding to metric of non-rotating body ($\vec{h} = 0$) is positive at any ξ for

$$\|\Delta\|^2(\xi) = \frac{1}{4}h^2\left[\frac{9}{4}(\xi+\mu)^2 - \frac{5}{4}m^2\right] \tag{48}$$

implies occurrence of superfluidity. Note that Eq. (48) does not duplicate the BCS gap (45) of pure scalar coupling but immediately reduces to

$$\|\Delta\|^2(\xi) = \frac{1}{4}h^2\left[m^2 + \frac{9}{2}\xi m\right] \approx \frac{1}{4}h^2 m^2 \tag{49}$$

in the non-relativistic limit $\mu \to m$. Indeed, the general formula (47) also tends to (45) for a non-relativistic system. It should be noted that the ultra-relativistic, or massless, Fermi-gas also has a non-zero gap

$$\|\Delta\|^2(\xi) = \frac{9}{16}h^2(\xi+\mu)^2 \tag{50}$$

due to tensor gravitational interaction.

6.2 Conclusion for a metric with rotation

While the gap corresponding to a pure spherical metric (48) does not convey the qualitative difference from the ordinary phenomenon of superfluidity with scalar coupling, the gravitational coupling whose metric includes rotation (finite $\vec{h}$) makes up the gap (47) depending both on momentum $|\vec{p}|$ (or ξ) and angle χ between $\vec{p}$ and $\vec{h}$. The latter dependence is a specific property of the gravitational coupling: it is not known in BCS with electromagnetic or scalar coupling. Meanwhile, we have not considered pairing with the non-zero orbital momentum (like in He-3) wherein one may expect new possibilities.

Eq. (47) implies that superfluidity may be forbidden at definite χ. The requirement of positive gap $\|\Delta\|^2(\xi,\chi)$ at arbitrary ξ and χ determines the condition

$$\frac{1}{4}h^2\left[\frac{9}{4}(\xi+\mu)^2 - \frac{5}{4}m^2\right] - \vec{h}^2\left[(\xi+\mu)^2 - m^2\right] > 0 \tag{51}$$

or

$$\left(\frac{9}{16}h^2 - \vec{h}^2\right)(\xi + \mu)^2 + \left(\vec{h}^2 - \frac{5}{16}h^2\right)m^2 > 0 \tag{52}$$

necessary for the occurrence of superfluidity. It is satisfied at arbitrary ξ (it can be negative) if

$$h^2 > \frac{16}{9}\vec{h}^2 \tag{53}$$

The same requirement is imposed on a massless field whose gap is

$$\|\Delta\|^2(\xi, \chi) = (\xi + \mu)^2 \left(\frac{9}{16}h^2 - \vec{h}^2 \cos^2\chi\right) \tag{54}$$

After all, we note that the gravitational coupling with metric tensor $h^\mu_\nu = \left\{h^k_0\right\}$ does not admit superfluidity at all.

The excitation spectrum obtained, can applied to the calculation of thermodynamic functions as it is used in the usual theory of superconductivity [10, 5], i.e. internal energy density, temperature dependence of gap etc.

7 Appendix: Equation of motion for a non-superfluid system

The equations of motion are

$$\left\{-i\beta\gamma^k \cdot \partial_k + \beta\left[m + \Sigma^{\#}\right]\right\} u_p = \epsilon_p u_p \tag{55}$$

$$\left\{-i\beta\gamma^k \cdot \partial_k + \beta\left[m + g_i\Gamma^i A^{\#}_i\right]\right\} \varphi_p(r) = \epsilon_p \varphi_p(r) \tag{56}$$

where we have used the notation $\hat{\psi}(r) = \sum_p \varphi_p(r)\,\hat{b}^\dagger_p$ with index $p = \{p\sigma\}$ related to single-particle baryon densities; also $\hat{\psi}(r) = \sum_p u_p(r)\,\hat{b}_p + v^*_p(r)\,\hat{b}^\dagger_p$. The self-consistent local potentials are defined as

$$A^{\#}_i(r) = A_i(r) + \frac{\delta R[n]}{\delta n^i(r)} \tag{57}$$

where the interacting energy $R[n] = E_H[n] + E_x[n] + E_c[n]$ includes the Hartree, exchange and correlation contribution. Particularly, $A^{\#}_H(r) = \int dt d^3 r_2 \Delta_i(t, r_2 - r_1)\, n^i(r_2)$.

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